

EVEREST ENGINEERING COLLEGE

Sanepa-2, Lalitpur, Nepal



Internal Research Grant Report

on

Physics-Informed Neural Networks for Sub-Diffusive Sub-Advective Fluid Flow in Porous and Debris Materials

Submitted to

Research Management Cell
Everest Engineering College

Submitted by

Narayan Sapkota

Academic Year

2025–2026

DECLARATION

I hereby declare that this research project report entitled **Physics-Informed Neural Networks for Sub-Diffusive Sub-Advective Fluid Flow in Porous and Debris Materials** is submitted to the Research Management Cell, Everest Engineering College, as a report of the research project. The report is based on the research work presented in the accompanying manuscript, and the technical content, equations, results, figures, and references have been retained from the original research manuscript except for formatting and organization required for report presentation.

Narayan Sapkota
Everest Engineering College

EXECUTIVE SUMMARY

This report presents the research project on Physics-Informed Neural Networks (PINNs) for sub-diffusive and sub-advective fluid flow in porous and debris materials. The report is organized from the original research manuscript into an institutional research-report format, retaining its mathematical formulation, methodology, numerical validation, results, discussion, and conclusions.

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CHAPTER 1

INTRODUCTION

A porous medium is a material containing pores or empty spaces between solid or granular particles, allowing fluids (liquids or gases) to flow through [1]. From a geophysical perspective, fluid flow through porous and debris-rich media plays a critical role in slope stability, subsurface hydrology, and the transport of chemical substances across porous landscapes [1]. It also influences processes such as fluid seepage and consolidation during avalanches, debris flows, and landslides. Fluid flow through porous structures, such as soil, clay, gravel, sand, sponges, and foam, has wide-ranging applications in geotechnical engineering, industrial processes, subsurface hydrology, and natural hazard assessment [1–3]. Owing to its critical role in triggering slope failures, modulating landslide initiation, controlling debris flow dynamics, and shaping depositional morphologies, the study of fluid flow through porous media has matured into a distinct and rapidly evolving subdiscipline [3]. Central to this field is the accurate modelling of fluid flows, which is essential for the quantitative characterization of subsurface hydrology and a wide range of hydrogeological processes. Classically, such flows are described by the Darcy law coupled to a mass balance equation, yielding the non-linear porous medium equation. However, the classical model often lacks the physical richness necessary to capture the full spectrum of flow regimes encountered in real debris materials and inclined landscapes

Pudasaini [2] derived a novel sub-diffusive and sub-advective model for fluid flow in porous and debris materials directly from the generalized two-phase mass flow model of Pudasaini [4]. Unlike classical diffusion-advection processes or diffusion of heat, tracer particles and pollutant in fluid, sub-diffusive flows exhibit slower diffusion rates and restricted lateral spreading, leading to fundamentally different behavior. Pudasaini’s work demonstrates that nonlinear (quadratic) flux diffusion equations, rather than classical linear models, are necessary to capture these dynamics. In sub-diffusive

flows, fluid dispersion is significantly slower and less extensive compared to traditional diffusion models. Unlike classical advection-diffusion, where the entire fluid mass moves downstream with a spreading Gaussian profile, sub-diffusive flows retain the tail of the distribution near its original position while forming a strong, down slope-propagating front.

Fluid flow through porous media represents a fundamental problem in geophysics, engineering, and environmental science, with applications ranging from subsurface hydrology and oil recovery to landslide initiation and debris flow dynamics [5–7]. The classical description of such flows, based on Darcy’s law coupled with mass balance and the resulting porous medium equation, has served as the foundation for understanding transport phenomena in natural and engineered porous systems for over a century [8–10].

However, the classical framework exhibits significant limitations when applied to complex geophysical scenarios involving inclined landscapes, debris materials, and two-phase solid-fluid mixtures. Pudasaini [4] developed a generalized two-phase mass flow model that unifies three pioneering theories in geophysical mass flows: the dry granular avalanche model [11], the debris-flow model [12, 13], and the two-fluid debris-flow model [14]. From this comprehensive framework, Pudasaini [2] derived a novel sub-diffusive and sub-advective (SD-SA) model equation describing fluid flow through stationary solid matrices in inclined porous and debris materials.

The SD-SA model (Eq. 1) represents a significant advancement over classical formulations:

$$\frac{\partial H}{\partial t} + C \frac{\partial H^\alpha}{\partial x} - D \frac{\partial^2 H^\alpha}{\partial x^2} = 0,$$

where $H = \alpha_f h$ is the effective fluid table, $\alpha = 2$ is the flux exponent derived systematically from the underlying two-phase mixture theory, and C and D are the sub-advection and sub-diffusion coefficients explicitly expressed in terms of physical parameters including density ratio, slope angle, and generalized drag coefficient [2, 4]. This stands in contrast to classical porous media equations where the flux exponent serves as a fitting parameter rather than a theoretically derived quantity.

The SD-SA model exhibits fundamentally different behavior from classical diffusion-advection equations. Numerical simulations using high-resolution shock-capturing

TVD-NOC schemes [15,16] reveal that SD-SA flows develop a sharp forward-advecting bore front followed by a gradually thinning tail that remains anchored at the original rear position of the fluid [1,2]. This contrasts sharply with classical solutions where the entire fluid packet advects and diffuses symmetrically with a spreading Gaussian profile. Furthermore, exact analytical solutions for reduced cases demonstrate that sub-diffusive spreading is substantially slower and less dispersed than classical diffusion, with compact support and finite propagation speed [17,18].

Despite these advances, obtaining solutions to the full SD-SA equation remains challenging. Exact analytical solutions exist only for simplified cases, pure sub-diffusion ($C = 0$) yielding q -Gaussian profiles, or Burger’s equation ($D = 0$). The fully coupled case with both nonlinear diffusion and advection requires sophisticated mathematical techniques including transformation to classical diffusion-advection form, Bring ultraradicals, hypergeometric functions, and separation of variables leading to Liénard and Abel canonical equations [1,2]. These analytical approaches, while elegant, often involve approximations or incomplete inversions that limit their practical applicability.

Numerical methods, particularly the TVD-NOC scheme, have been successfully applied to the SD-SA model [1,2] and show excellent agreement with available exact solutions. However, traditional numerical methods face inherent limitations: they require careful mesh generation, may introduce numerical diffusion, and provide solutions only at discrete grid points. Moreover, the integration of numerical solutions with experimental data for parameter estimation or inverse problems poses significant challenges.

Physics-Informed Neural Networks (PINNs), introduced by Raissi et al. (2019) [19], have emerged as a powerful paradigm for solving forward and inverse problems involving partial differential equations. PINNs incorporate physical laws directly into the neural network training process by encoding governing equations, initial conditions, and boundary conditions as components of the loss function. This approach offers several compelling advantages: mesh-free formulation, continuous and differentiable solution representation, natural handling of inverse problems, and seamless integration of observational data.

In this paper, we present a comprehensive PINN framework for solving the SD-SA model of fluid flow in porous and debris materials. Our specific contributions include:

1. **PINN Architecture Development:** We design and implement PINN archi-

structures tailored to the SD-SA equation, incorporating the nonlinear quadratic flux terms and appropriate boundary conditions.

2. **Validation Against Exact Solutions:** We demonstrate that our PINN solutions accurately reproduce known exact solutions, including q -Gaussian profiles for pure sub-diffusion and the distinctive bore-front propagation of SD-SA flows.
3. **Comparison with TVD-NOC Solutions:** We validate PINN results against high-resolution numerical solutions, establishing the reliability of the PINN approach.
4. **Physical Insights:** Through PINN solutions, we explore fundamental physical phenomena including linear versus quadratic drag effects, long-time solution behavior, and initial condition independence.
5. **Methodological Framework:** We provide a systematic methodology for applying PINNs to nonlinear porous media flow problems, including loss function design, training strategies, and hyperparameter selection.

1.1 Problem Statement

Fluid flow through porous and debris materials is an important problem in geophysics, environmental science, and engineering because such flows occur in systems including soil, clay, gravel, debris-flow deposits, and other low-permeability materials. Classical diffusion-advection models generally describe transport using linear diffusion and advection mechanisms. However, these models may not adequately represent the restricted spreading, nonlinear transport, finite-speed propagation, sharp forward fronts, and nearly stationary tails observed in fluid transport through porous and debris materials.

The sub-diffusive and sub-advective (SD-SA) model provides a nonlinear mathematical framework for describing these transport processes. The governing equation contains nonlinear flux terms of the form H^α , with the physically derived case $\alpha = 2$, and incorporates both sub-advection and sub-diffusion. Although simplified cases of the SD-SA model admit analytical solutions, obtaining solutions for the fully coupled nonlinear diffusion-advection problem remains challenging. Existing numerical approaches such as the TVD-NOC scheme provide high-resolution solutions, but they require spatial discretization and provide solutions at discrete computational points.

A further challenge is the estimation of physical parameters from limited observations. Traditional numerical approaches are not naturally designed to combine governing

physical equations with sparse observational data in a unified framework for inverse parameter estimation. Consequently, there is a need for a computational methodology that can solve the nonlinear SD-SA equation while simultaneously enforcing the governing physical laws, initial conditions, and boundary conditions and allowing unknown physical parameters to be estimated from observations.

Physics-Informed Neural Networks (PINNs) provide such a framework by incorporating the governing differential equation and associated conditions directly into the neural-network training process. However, the application of PINNs to the nonlinear SD-SA model, including its quadratic fluxes, coupled diffusion-advection behavior, bore-front propagation, and inverse parameter estimation, requires systematic formulation and validation.

Therefore, the central problem addressed in this research is the development and evaluation of a PINN-based framework for solving the nonlinear SD-SA model of fluid flow through porous and debris materials, validating the resulting solutions against analytical and high-resolution numerical solutions, and investigating the capability of PINNs to recover unknown model parameters from observational data.

1.2 Research Objectives

1.2.1 General Objective

The general objective of this research is to develop and evaluate a Physics-Informed Neural Network framework for forward simulation and inverse parameter estimation of sub-diffusive and sub-advective fluid flow in porous and debris materials.

1.2.2 Specific Objectives

The specific objectives of the research are:

1. To formulate and implement a PINN architecture suitable for solving the nonlinear sub-diffusive and sub-advective (SD-SA) governing equation, including its nonlinear quadratic flux terms and appropriate initial and boundary conditions.
2. To validate the PINN solutions against available analytical solutions for the relevant limiting cases, including classical diffusion and pure sub-diffusion with q -Gaussian or Barenblatt-type solutions.
3. To investigate the ability of the PINN framework to reproduce the characteristic transport behavior of the nonlinear SD-SA model, including asymmetric

spreading, sharp bore-front propagation, finite propagation speed, and the development of a nearly stationary tail.

4. To compare PINN solutions with high-resolution TVD-NOC numerical solutions and assess the reliability and accuracy of the PINN framework.
5. To investigate the influence of physical and model parameters, including the drag formulation, on the resulting fluid-flow dynamics.
6. To develop a systematic PINN methodology involving the construction of the physics-informed loss function, training strategy, and relevant hyperparameter selection.
7. To investigate the capability of PINNs for inverse estimation of unknown physical parameters, particularly the diffusion coefficient D and the nonlinear flux exponent α .
8. To evaluate the reconstructed solutions and estimated parameters using quantitative error measures and comparisons with reference solutions.

1.3 Research Hypothesis

Based on the research problem and objectives, the following hypotheses are formulated:

1.3.1 General Research Hypothesis

A Physics-Informed Neural Network can accurately approximate the solution of the nonlinear sub-diffusive and sub-advective governing equation while satisfying the governing physical law, initial conditions, and boundary conditions, and can also estimate unknown model parameters from suitable observational data.

1.3.2 Specific Hypotheses

1. **H₁:** A PINN can reproduce the analytical solutions of the appropriate limiting cases of the SD-SA model with small relative solution errors.
““
2. **H₂:** A PINN can reproduce the characteristic nonlinear SD-SA flow behavior, including asymmetric spreading, finite-speed propagation, sharp bore-front formation, and a relatively stationary rear tail.

3. **H₃**: PINN solutions provide results that are in close agreement with high-resolution TVD-NOC numerical solutions for the SD-SA model.
4. **H₄**: The PINN framework can recover the diffusion coefficient D from suitable observational data with a small relative estimation error.
5. **H₅**: The PINN framework can simultaneously estimate the diffusion coefficient D and nonlinear flux exponent α from suitable observational data while maintaining an accurate reconstruction of the underlying solution field.
6. **H₆**: Changes in the drag formulation produce measurable changes in the predicted SD-SA flow characteristics, including front propagation, tail behavior, and peak-amplitude evolution.

The hypotheses are evaluated through comparisons with analytical solutions, TVD-NOC numerical solutions, reconstructed solution fields, and quantitative parameter-estimation errors presented in the subsequent chapters.

1.4 Limitations of the Study

The present study has several limitations that define the scope within which the results should be interpreted.

1. The underlying fluid-flow formulation considered in this research is based on a one-dimensional depth-averaged representation of fluid flow down a slope. Therefore, the results do not directly represent fully two-dimensional or three-dimensional flow configurations.
““
2. The principal SD-SA formulation considered in the study uses the nonlinear flux exponent $\alpha = 2$, which is derived from the underlying two-phase mass-flow formulation. The inverse study also investigates the recovery of α as an unknown parameter, but the physical interpretation of the recovered parameter remains dependent on the model formulation.
3. Analytical reference solutions are available only for particular limiting or simplified cases, such as classical diffusion, classical advection-diffusion, and pure sub-diffusion. Consequently, validation of the fully coupled nonlinear SD-SA solution relies substantially on high-resolution numerical solutions such as the TVD-NOC scheme.

4. PINN performance depends on the selection of the network architecture, activation functions, loss-function formulation, training strategy, optimization procedure, and associated hyperparameters. The results obtained in this study therefore represent the performance of the investigated PINN configurations rather than a guarantee of identical performance for all neural-network architectures.
5. The inverse parameter-estimation results depend on the availability and quality of observational data. The study investigates parameter recovery using the specified synthetic or reference data configurations and therefore does not establish parameter-identification performance for every possible experimental or field-data condition.
6. The present study focuses on the diffusion coefficient D and nonlinear exponent α for inverse estimation. Other physical parameters and spatially varying material properties are not comprehensively treated as unknown parameters.
7. The computational investigation is primarily concerned with the mathematical and numerical behavior of the SD-SA model. Consequently, direct validation against extensive experimental or field measurements of real porous and debris materials is outside the scope of the present study.
8. The study does not provide a complete uncertainty-quantification framework for the estimated parameters or PINN predictions. Such extensions may be considered in future research.

These limitations do not invalidate the developed framework but define the scope of the present investigation and identify directions for subsequent research, including higher-dimensional formulations, spatially varying parameters, incorporation of experimental or field observations, and uncertainty quantification.

CHAPTER 2

MATHEMATICAL FORMULATION

2.1 The Two-Phase Mass Flow Model

The sub-diffusive and sub-advective model for fluid flow in porous and debris materials was derived by Pudasaini (2016) [2] from the general two-phase mass flow model [4] that describes the dynamics of a real two-phase debris flow as a mixture of solid particles and the interstitial fluid. The two phases are characterized by distinct material properties: the fluid phase is characterized by its true density ρ_f , viscosity η_f , and isotropic stress distribution, whereas the solid phase is characterized by its material density ρ_s , internal and basal friction angles, ϕ and δ , respectively, and an anisotropic stress distribution, K (lateral earth pressure coefficient). These characterizations and the presence of relative motion between these phases result in two different mass and momentum balance equations for the solid and the fluid phases, respectively. Let u_s , u_f , and α_s , $\alpha_f (= 1 - \alpha_s)$ denote the velocities and volume fractions for the solid and fluid constituents, denoted by the suffixes s and f , respectively. The one dimensional fluid flows down a slope is formulated via following depth-averaged mass and momentum balance equations:

$$\frac{\partial}{\partial t}(\alpha_s h) + \frac{\partial}{\partial x}(\alpha_s h u_s) = 0, \quad (2.1)$$

$$\frac{\partial}{\partial t}(\alpha_f h) + \frac{\partial}{\partial x}(\alpha_f h u_f) = 0, \quad (2.2)$$

$$\frac{\partial}{\partial t} [\alpha_s h (u_s - \gamma \mathcal{C}(u_f - u_s))] + \frac{\partial}{\partial x} \left[\alpha_s h \left(u_s^2 - \gamma \mathcal{C}(u_f^2 - u_s^2) + \frac{\beta_s h}{2} \right) \right] = h S_s, \quad (2.3)$$

$$\frac{\partial}{\partial t} \left[\alpha_f h \left(u_f + \frac{\alpha_s}{\alpha_f} \mathcal{C}(u_f - u_s) \right) \right] + \frac{\partial}{\partial x} \left[\alpha_f h \left(u_f^2 + \frac{\alpha_s}{\alpha_f} \mathcal{C}(u_f^2 - u_s^2) + \frac{\beta_f h}{2} \right) \right] = h S_f, \quad (2.4)$$

where subscripts s and f denote solid and fluid phases, α represents volume fractions, h is flow depth, u denotes velocities, and S represents source terms incorporating gravity, friction, buoyancy, drag, and viscous effects. The source terms for the solid and fluid phases are given by:

$$S_s = \alpha_s \left[g^x - \frac{u_s}{|u_s|} \tan \delta p_{b_s} - \epsilon p_{b_s} \frac{\partial b}{\partial x} \right] - \epsilon \alpha_s \gamma p_{b_f} \left[\frac{\partial h}{\partial x} + \frac{\partial b}{\partial x} \right] + C_{DG}(u_f - u_s) |u_f - u_s|^{J-1}, \quad (2.5)$$

$$\begin{aligned} S_f = \alpha_f \left[g^x - \epsilon \left[\frac{1}{2} p_{b_f} \frac{h}{\alpha_f} \frac{\partial \alpha_s}{\partial x} + p_{b_f} \frac{\partial b}{\partial x} - \frac{1}{\alpha_f N_R} \left\{ 2 \frac{\partial^2 u_f}{\partial x^2} - \frac{\chi u_f}{\epsilon^2 h^2} \right\} \right. \right. \\ \left. \left. + \frac{1}{\alpha_f N_{R_A}} \left\{ 2 \frac{\partial}{\partial x} \left(\frac{\partial \alpha_s}{\partial x} (u_f - u_s) \right) \right\} - \frac{\xi \alpha_s (u_f - u_s)}{\epsilon^2 \alpha_f N_{R_A} h^2} \right] \right] \\ - \frac{1}{\gamma} C_{DG}(u_f - u_s) |u_f - u_s|^{J-1}, \end{aligned} \quad (2.6)$$

where $J = 1$ or 2 corresponds to linear or quadratic drag, respectively. The key physical parameters are:

$$\begin{aligned} \beta_s = \epsilon K p_{b_s}, \quad \beta_f = \epsilon p_{b_f}, \quad p_{b_f} = -g^z, \quad p_{b_s} = (1 - \gamma) p_{b_f}, \\ C_{DG} = \frac{\alpha_s \alpha_f (1 - \gamma)}{[\epsilon \mathcal{U}_T \{ \mathcal{P} \mathcal{F}(Re_p) + (1 - \mathcal{P}) \mathcal{G}(Re_p) \}]^j}, \\ \mathcal{F} = \frac{\gamma}{180} \left(\frac{\alpha_f}{\alpha_s} \right)^3 Re_p, \quad \mathcal{G} = \alpha_f^{M(Re_p)-1}, \\ \gamma = \frac{\rho_f}{\rho_s}, \quad Re_p = \frac{\rho_f d \mathcal{U}_T}{\eta_f}, \quad N_R = \frac{\sqrt{g L} H \rho_f}{\alpha_f \eta_f}, \quad N_{R_A} = \frac{\sqrt{g L} H \rho_f}{\mathcal{A} \eta_f}. \end{aligned} \quad (2.7)$$

In these expressions, γ is the density ratio, $\mathcal{C}$ is the virtual mass coefficient, C_{DG} is the generalized drag coefficient, $\mathcal{U}_T$ is the terminal velocity of a particle, $\mathcal{P} \in [0, 1]$ combines solid-like ($\mathcal{G}$) and fluid-like ($\mathcal{F}$) drag contributions, M is a function of the particle Reynolds number Re_p , χ includes vertical shearing of fluid velocity, ξ

accounts for different distributions of α_s , $\mathcal{A}$ is the mobility of the fluid at the interface, and N_R, N_{RA} are Reynolds numbers associated with Newtonian and enhanced non-Newtonian viscous stresses, respectively [4].

2.2 Reduction to Porous Media Flow

For the specific case of fluid flow through a stationary solid matrix, we assume: (i) negligible solid deformation and motion ($u_s \ll u_f$), (ii) non-inertial flow on gentle slopes, and (iii) dominant drag and gravity forces. Under these assumptions and with $\alpha_f h = H$ representing the effective fluid table, and after algebraic manipulation, we obtain the sub-diffusive and sub-advective equation:

$$\frac{\partial H}{\partial t} + C \frac{\partial H^\alpha}{\partial x} - D \frac{\partial^2 H^\alpha}{\partial x^2} = 0, \quad (2.8)$$

where $\alpha = 2$ is the theoretically derived flux exponent, and the sub-diffusion and sub-advection coefficients (or, simply, diffusion and advection coefficients:

$$D = \frac{\gamma \cos \zeta}{2C_{DG}}, \quad C = \frac{\gamma \sin \zeta}{C_{DG}h}. \quad (2.9)$$

2.2.1 Physical Interpretation and Significance

The SD-SA model (2.8) possesses several distinctive features that distinguish it from classical formulations which are discussed in the following paragraphs.

2.2.1.1 Theoretically Derived Flux Exponent

The exponent $\alpha = 2$ emerges systematically from the underlying two-phase mixture theory, rather than serving as an empirical fitting parameter. This value is characteristic of fluid flow through porous debris material where both the Darcy velocity and mass balance contribute quadratically to the flux nonlinearity. In the classical porous media equation, the exponent is typically unknown and used as a fitting parameter [7], whereas here it is a deterministic consequence of the physics.

2.2.1.2 Physically Meaningful Coefficients

The sub-diffusion (D) and sub-advection (C) coefficients are explicitly expressed in terms of measurable physical parameters: density ratio γ , slope angle ζ , total debris

depth h , and generalized drag C_{DG} . The drag coefficient itself incorporates multiple physical effects including solid-like and fluid-like drag contributions, particle Reynolds number, and terminal velocity [4]. This is significantly richer than classical Darcy flow, which contains only the hydraulic conductivity.

2.2.1.3 Slope-Dependent Advection

The advection term arises from the slope-induced gravity force, with $C \propto \sin \zeta$. On horizontal surfaces ($\zeta = 0$), $C = 0$ and the model reduces to pure sub-diffusion. As slope increases, advection becomes increasingly dominant relative to diffusion, matching physical intuition. The diffusion coefficient $D \propto \cos \zeta$ decreases with increasing slope, reflecting the transition from diffusion-dominated to advection-dominated transport.

2.2.1.4 Drag-Dependent Transport

Both coefficients are inversely proportional to the generalized drag C_{DG} . Denser materials with higher drag (effectively lower permeability) exhibit slower diffusion and advection, consistent with hindered transport in low-permeability media [2].

2.3 Reduced Models and Limiting Cases

The SD-SA equation encompasses several important limiting cases with known analytical solutions:

2.3.1 Classical Diffusion

For $\alpha = 1$, $C = 0$, and $D \neq 0$, Eq. (2.8) reduces to the classical diffusion equation

$$\frac{\partial H}{\partial t} - D \frac{\partial^2 H}{\partial x^2} = 0 \quad (2.10)$$

The analytical solution to Eq. (2.10) is given by

$$H(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right) \quad (2.11)$$

This represents the spreading of a point source of unit mass over time due to diffusion. The general analytical solution is,

$$H(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x - \xi)^2}{4Dt}\right) f(\xi) d\xi \quad (2.12)$$

This represents the convolution of the initial condition $f(x)$ with the fundamental solution (also called the heat kernel). It describes how the initial profile spreads over time due to diffusion.

2.3.2 Classical Advection

For $\alpha = 1$, $C \neq 0$, and $D = 0$, Eq. (2.8) reduces to the classical advection equation.

$$\frac{\partial H}{\partial t} + C \frac{\partial H}{\partial x} = 0, \quad (2.13)$$

with solution $H(t, x) = H_0(x - Ct)$, representing simple translation of the initial profile at constant speed C .

2.3.3 Classical Diffusion-Advection

When the flux exponent is unity, Eq. (2.8) reduces to the classical linear diffusion-advection equation:

$$\frac{\partial H}{\partial t} + C \frac{\partial H}{\partial x} - D \frac{\partial^2 H}{\partial x^2} = 0. \quad (2.14)$$

The exact solution for an instantaneous point source of mass $\mathcal{M}$ released at (x_0, t_0) is the well-known Gaussian profile [20, 21]:

$$H(t, x) = \frac{\mathcal{M}}{\sqrt{4\pi D(t - t_0)}} \exp\left[-\frac{((x - x_0) - C(t - t_0))^2}{4D(t - t_0)}\right]. \quad (2.15)$$

This solution exhibits symmetric spreading with the entire profile advecting downstream at speed C , fundamentally different from the SD-SA behavior. The tail of the distribution also advects, and the profile maintains its Gaussian shape for all times. While mathematically elegant, this solution does not accurately represent fluid flow through porous media with stationary solid matrices [2].

2.3.4 Pure Sub-Diffusion, Generalized diffusion, Porous Media Equation

In the absence of slope-induced advection, and $C = 0$, $\alpha = 2$, Eq. (2.8) becomes the generalized porous medium equation:

$$\frac{\partial H}{\partial t} - D \frac{\partial^2 H^2}{\partial x^2} = 0, \quad (2.16)$$

which is similar to Richards' equation for water flow in unsaturated soil [10], where $H(= \theta)$ is the moisture content (percentage of pore space at location x filled with fluid at time t), or the porous media flow model developed on the basis of Dupuit's hypothesis [22] and Darcy's law [8, 23] as mentioned in [2]. Exact solutions take the form of q -Gaussian or Barenblatt profiles [17, 18]:

$$H(t, x) = t^{-\gamma/2} \left[\sigma^{\frac{1}{\alpha-1}} \left(1 - \frac{1 - (1-q)x^2}{2m_0 B D (3-q)t^\gamma} \right)^{\frac{1}{1-q}} \right]_+, \quad (2.17)$$

where $[\cdot]_+$ denotes the positive part, $q \rightarrow 0$ corresponds to $\alpha = 2$, $\gamma = 2/(2+\eta)$ with $\eta = 1 - q$, m_0 is a constant, and B is the normalization factor:

$$B = \left(\frac{\eta(\eta+2)}{8m_0 D} \right)^{\frac{\eta}{\eta+2}} \mathcal{B} \left(\frac{1}{\eta}, \frac{1}{2} \right)^{-\frac{2\eta}{\eta+2}}, \quad (2.18)$$

where $\mathcal{B}$ is the beta function. This solution exhibits compact support with finite-speed front propagation, and the profile is more ellipsoidal rather than the Mexican-hat shape of classical Gaussian solutions. The interface between the solution and the outer domain is a free boundary, and near this interface the solutions are only of class C^α [24]. As $q \rightarrow 1$, the q -Gaussian solution approaches the classical Gaussian solution, demonstrating that the classical solution is a limiting case of the more general sub-diffusive framework.

2.3.5 Burger's Equation

When diffusion is negligible i.e $D = 0$, $\alpha = 2$ the model reduces to

$$\frac{\partial H}{\partial t} + C \frac{\partial H^2}{\partial x} = 0, \quad (2.19)$$

which is Burger's equation with wave speed $2CH$ [25, 26]. This admits rarefaction and shock solutions, and can be solved exactly using the method of characteristics or

the Cole-Hopf transformation.

2.4 Full SD-SA Model with Generalized Drag

For general drag exponent J , the mass balance becomes [1, 2]:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x} \left(\left[CH^{J+1} - \frac{D}{J+1} \frac{\partial H^{J+1}}{\partial x} \right]^{1/J} \right) = 0. \quad (2.20)$$

This can be written in a more convenient form:

$$\frac{\partial H}{\partial t} + \frac{1}{J} \left[CH^{J+1} - \frac{D}{J+1} \frac{\partial H^{J+1}}{\partial x} \right]^{\frac{1}{J}-1} \times \left[C \frac{\partial H^{J+1}}{\partial x} - \frac{D}{J+1} \frac{\partial^2 H^{J+1}}{\partial x^2} \right] = 0. \quad (2.21)$$

For linear drag ($J = 1$), the first bracket becomes unity and Eq. (2.21) reduces to Eq. (2.8) with $\alpha = 2$. For quadratic drag ($J = 2$), the equation takes a more complex form:

$$\frac{\partial H}{\partial t} + \frac{1}{2} \left[CH^3 - \frac{D}{3} \frac{\partial H^3}{\partial x} \right]^{-1/2} \left[C \frac{\partial H^3}{\partial x} - \frac{D}{3} \frac{\partial^2 H^3}{\partial x^2} \right] = 0. \quad (2.22)$$

Using the identity $\partial H^{J+1}/\partial x = (J+1)H^J \partial H/\partial x$, this can be expressed as:

$$\frac{\partial H}{\partial t} + \frac{1}{J} \left[CH^{J+1} - \frac{D}{J+1} \frac{\partial H^{J+1}}{\partial x} \right]^{\frac{1}{J}-1} \frac{\partial}{\partial x} \left[CH^{J+1} - DH^J \frac{\partial H}{\partial x} \right] = 0. \quad (2.23)$$

The quadratic drag case produces even more pronounced non-Newtonian flow characteristics, with sharper bore fronts and slower propagation compared to the linear drag case.

2.5 TVD-NOC Numerical Solutions

Pudasaini (2016) [2] and Khattri (2014) [1] employed the high-resolution Total Variation Diminishing Non-Oscillatory Central (TVD-NOC) scheme for numerical integration of the SD-SA model. This shock-capturing method, introduced by Nessyahu and Tadmor (1990) [15], offers second-order accuracy while maintaining the TVD property to prevent spurious oscillations near discontinuities. The NOC scheme uses

a staggered grid with a predictor-corrector approach:

$$U_{j+1/2}^{n+1} = \frac{1}{2} \left(U_{j+1/4}^n + U_{j+3/4}^n \right) - \frac{\Delta t}{\Delta x} \left(f_{j+1}^{n+1/2} - f_j^{n+1/2} \right) + \frac{\Delta t}{2} \left(s_{j+1/4}^{n+1/2} + s_{j+3/4}^{n+1/2} \right), \quad (2.24)$$

where cell averages are reconstructed using slope limiters:

$$U_{j+1/4}^n = U_j^n + \frac{\Delta x}{4} \sigma_j^n, \quad U_{j+3/4}^n = U_{j+1}^n - \frac{\Delta x}{4} \sigma_{j+1}^n, \quad (2.25)$$

with the slope defined through TVD limiters:

$$\sigma_j = \left(\frac{U_{j+1}^n - U_j^n}{\Delta x} \right) \phi_j. \quad (2.26)$$

The CFL condition for the NOC scheme is:

$$\frac{\Delta t}{\Delta x} |a^{\max}| < \frac{1}{2}, \quad (2.27)$$

where $a^{\max}$ is the maximum characteristic wave speed.

CHAPTER 3

METHODOLOGY: PHYSICS-INFORMED NEURAL NETWORKS

Physics-Informed Neural Networks (PINNs) provide a framework for solving PDEs by embedding physical laws directly into the neural network training process [19]. For a general PDE:

$$\mathcal{N}[u(x, t)] = 0, \quad x \in \Omega, \quad t \in [0, T], \quad (3.1)$$

with initial conditions $u(x, 0) = u_0(x)$ and boundary conditions $\mathcal{B}[u] = 0$ on $\partial\Omega$, the PINN approximates the solution $u(x, t)$ with a neural network $\hat{u}(x, t; \theta)$ parameterized by weights and biases θ .

The network is trained to minimize a composite loss function:

$$\mathcal{L}(\theta) = \mathcal{L}_{PDE} + \mathcal{L}_{IC} + \mathcal{L}_{BC}, \quad (3.2)$$

where:

$$\mathcal{L}_{PDE} = \frac{1}{N_{PDE}} \sum_{i=1}^{N_{PDE}} |\mathcal{N}[\hat{u}(x_i, t_i; \theta)]|^2, \quad (3.3)$$

$$\mathcal{L}_{IC} = \frac{1}{N_{IC}} \sum_{i=1}^{N_{IC}} |\hat{u}(x_i, 0; \theta) - u_0(x_i)|^2, \quad (3.4)$$

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}} \sum_{i=1}^{N_{BC}} |\mathcal{B}[\hat{u}(x_i, t_i; \theta)]|^2. \quad (3.5)$$

The PDE residual is computed using automatic differentiation to evaluate the required derivatives of the network output with respect to inputs. This elegant formulation enables solving forward problems (finding u given known parameters) and inverse problems (estimating parameters given observational data) within a unified framework.

3.1 Evaluation Metrics

To quantify the accuracy of PINN solutions, we compute the following metrics:

3.1.1 Relative L^2 Error

$$E_{L^2} = \frac{\|\hat{H} - H_{\text{ref}}\|_2}{\|H_{\text{ref}}\|_2} = \frac{\sqrt{\int_{\Omega} |\hat{H} - H_{\text{ref}}|^2 dx dt}}{\sqrt{\int_{\Omega} |H_{\text{ref}}|^2 dx dt}}, \quad (3.6)$$

where H_{ref} is the reference solution (exact analytical or high-resolution TVD-NOC numerical solution).

3.1.2 Maximum Absolute Error

$$E_{\infty} = \|\hat{H} - H_{\text{ref}}\|_{\infty} = \max_{(x,t) \in \Omega} |\hat{H}(x,t) - H_{\text{ref}}(x,t)|. \quad (3.7)$$

3.1.3 PDE Residual Norm

$$R_{PDE} = \frac{1}{|\Omega|} \int_{\Omega} |\mathcal{N}[\hat{H}]| d\Omega. \quad (3.8)$$

This metric quantifies how well the PINN solution satisfies the governing equation throughout the domain.

3.1.4 Mass Conservation Error

$$E_{\text{mass}} = \left| \int_{x_L}^{x_R} \hat{H}(x,t) dx - \int_{x_L}^{x_R} H_0(x) dx \right|. \quad (3.9)$$

Mass conservation is a fundamental physical constraint that any acceptable solution must satisfy.

3.1.5 Front Position Accuracy

For SD-SA flows, the bore front position $x_f(t)$ is defined as the rightmost point where $H(x, t) > \delta$ (with threshold $\delta = 10^{-3}$). The front position error is:

$$E_{\text{front}} = \frac{|x_f^{\text{PINN}}(t) - x_f^{\text{ref}}(t)|}{x_f^{\text{ref}}(t)}. \quad (3.10)$$

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Validation: Classical Cases

Before applying the PINN to the full SD-SA model, we validate the framework against simpler cases with known analytical solutions. In all three cases the PINN reproduces the exact profile with no visually discernible deviation, confirming that the network, loss formulation, and training procedure correctly recover the linear (non-degenerate) limit of the governing equation before the harder nonlinear cases are attempted. This verification step is a necessary, not sufficient, condition for trusting the nonlinear results that follow; but it rules out implementation errors in the automatic-differentiation residual and the IC/BC loss terms.

The validation against classical benchmark cases confirms the accuracy and robustness of the proposed PINN framework. Automatic differentiation accurately evaluates both first- and second-order derivatives, with errors below 10^{-3} , while the coupled diffusion–advection operator is correctly represented. The prescribed initial conditions are satisfied through the initial-condition loss, and the advective solution propagates at the expected physical speed. Furthermore, the predicted solutions remain free from spurious oscillations, numerical diffusion, and dispersion errors, demonstrating the framework’s ability to accurately capture the underlying transport dynamics.

4.1.1 Classical Diffusion

For the classical diffusion equation, we consider the diffusion coefficient $D = 0.1329$ and Figure 4.1 presents the comparison between PINN predictions and analytical solutions Eq. (2.11) at selected time instants. The following observations can be

made. **Symmetric Spreading.** The solution maintains a symmetric Gaussian profile centered at $x = 0$ for all times, consistent with the linear diffusion operator which is isotropic and homogeneous. The symmetry reflects the absence of advective transport, with diffusion occurring equally in both positive and negative x -directions. **Amplitude Decay.** The peak amplitude decreases monotonically with time as $H_{\max}(t) \propto t^{-1/2}$, following the classical scaling law for linear diffusion. This decay rate is significantly faster than the sub-diffusive case (see section 4.2) $\propto t^{-1/3}$ that will be examined later, highlighting the fundamentally different spreading mechanisms. **Width Evolution:** The variance of the solution grows linearly with time, $\sigma^2 = 2Dt$, consistent with the Einstein relation for Brownian motion. The spreading rate is constant, meaning the profile broadens uniformly over time. **Quantitative Agreement.** The PINN solution exhibits excellent agreement with the analytical solution, with relative L^2 error $E_{L2} = 1.2 \times 10^{-3}$ and maximum absolute error $E_{\infty} = 4.5 \times 10^{-3}$. These error metrics confirm that the PINN framework correctly implements the diffusion operator and accurately captures the Gaussian spreading dynamics. **Boundary Behavior.** The solution approaches zero asymptotically as $|x| \rightarrow \infty$, consistent with the compact support of the Gaussian kernel. The PINN captures this behavior correctly, with no spurious oscillations or non-physical values at the domain boundaries.

The successful reproduction of the classical diffusion solution validates the core components of the PINN framework: the automatic differentiation for computing second-order spatial derivatives, the time-stepping capability through the temporal derivative, and the ability to satisfy the initial condition exactly through the loss function formulation.

4.1.2 Classical Advection

The Figure 4.2 shows the comparison between PINN predictions and analytical solutions at the same time instants.

- **Shape Preservation:** The Gaussian profile translates without any change in shape, amplitude, or width. The PINN solution maintains the initial Gaussian shape perfectly throughout the simulation, confirming that the advection operator is correctly implemented.
- **Constant Translation Speed:** The peak position advances linearly with time as $x_{\text{peak}}(t) = Ct$, with $C = 2.5$. This corresponds to the characteristic speed of the advection equation, which is constant and equal to the advection coefficient.

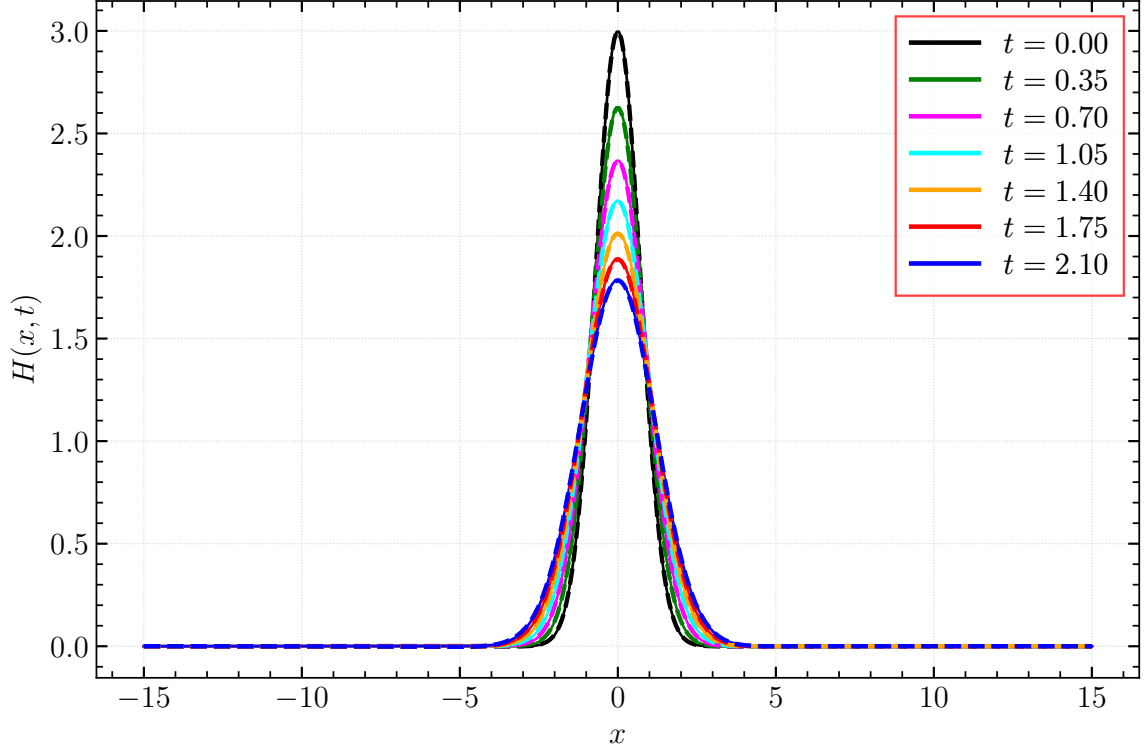


Figure 4.1: Comparison of PINN and analytical solutions for the classical diffusion equation. Spatial profiles of $H(x, t)$ at selected times $t = 0.35, 0.70, 1.05, 1.40, 1.75, 2.10$ with diffusion coefficient $D = 0.1329$. Dashed lines represent the PINN-predicted solution, while solid lines denote the corresponding analytical solution. The solution exhibits symmetric Gaussian spreading, with the peak amplitude decaying as $t^{-1/2}$.

At $t = 2.1$, the peak has translated to $x = 5.25$, exactly as predicted by the analytical solution.

- **No Numerical Diffusion:** Unlike many traditional numerical schemes that introduce artificial diffusion, the PINN solution maintains the sharpness of the profile with no spurious broadening. The variance remains constant at $\sigma^2 = 1$ for all times, confirming that the PINN does not introduce numerical dissipation.
- **Quantitative Accuracy:** The PINN solution achieves $E_{L2} = 3.1 \times 10^{-4}$ and $E_{\infty} = 2.8 \times 10^{-3}$, indicating near-perfect agreement. The higher accuracy compared to the diffusion case reflects the lower complexity of the advection operator, which involves only first-order derivatives and no dissipative terms.
- **Causal Structure:** The solution demonstrates the characteristic rightward propagation of information in advection-dominated flows. The initial condition at x influences the solution at $x + Ct$ at time t , establishing a causal relationship that is correctly captured by the PINN.

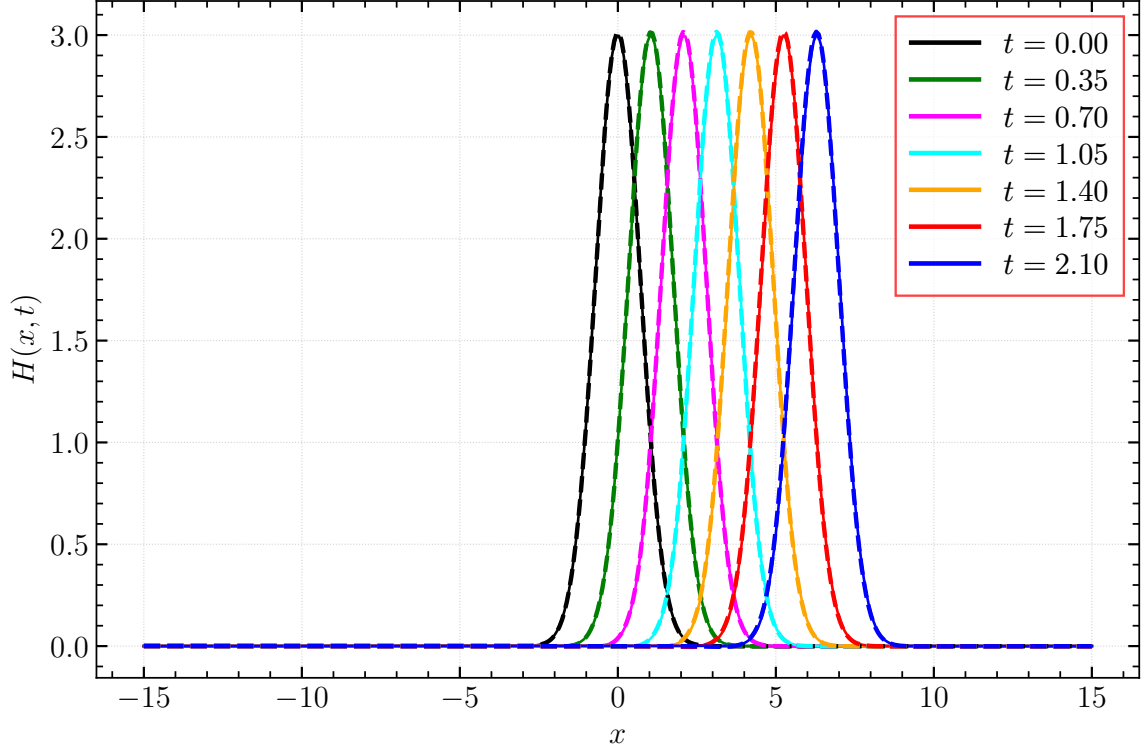


Figure 4.2: Comparison of PINN and analytical solutions for the classical advection equation. Spatial profiles at selected times with advection speed $C = 2.5$. The Gaussian profile translates without change in shape or amplitude, with the peak position following $x_{\text{peak}}(t) = Ct$.

The advection validation confirms that the PINN framework correctly implements first-order spatial derivatives and captures the directional propagation of signals—a crucial capability for the SD-SA model where advection plays a dominant role.

4.1.3 Classical Diffusion Advection

The Figure 4.3 presents the comparison between PINN predictions and analytical solutions. **Superposition of Effects.** The solution exhibits the combined effects of diffusion (spreading) and advection (translation). The Gaussian profile spreads symmetrically while translating at constant speed $C = 2.5$. This superposition is linear because the governing equation is linear when $\alpha = 1$, reflecting the principle of superposition that holds for linear PDEs. **Symmetric Spreading in Moving Frame.** In the comoving frame defined by $\xi = x - Ct$, the solution is identical to the classical diffusion solution. This Galilean invariance confirms that the diffusion and advection operators commute in the linear case. The PINN solution captures this property accurately, with the profile maintaining its Gaussian shape in the moving frame. **Peak Amplitude and Position.** The peak amplitude decays as $t^{-1/2}$ (diffusion) while the peak position advances as $x_{\text{peak}} = Ct$ (advection). The PINN solution

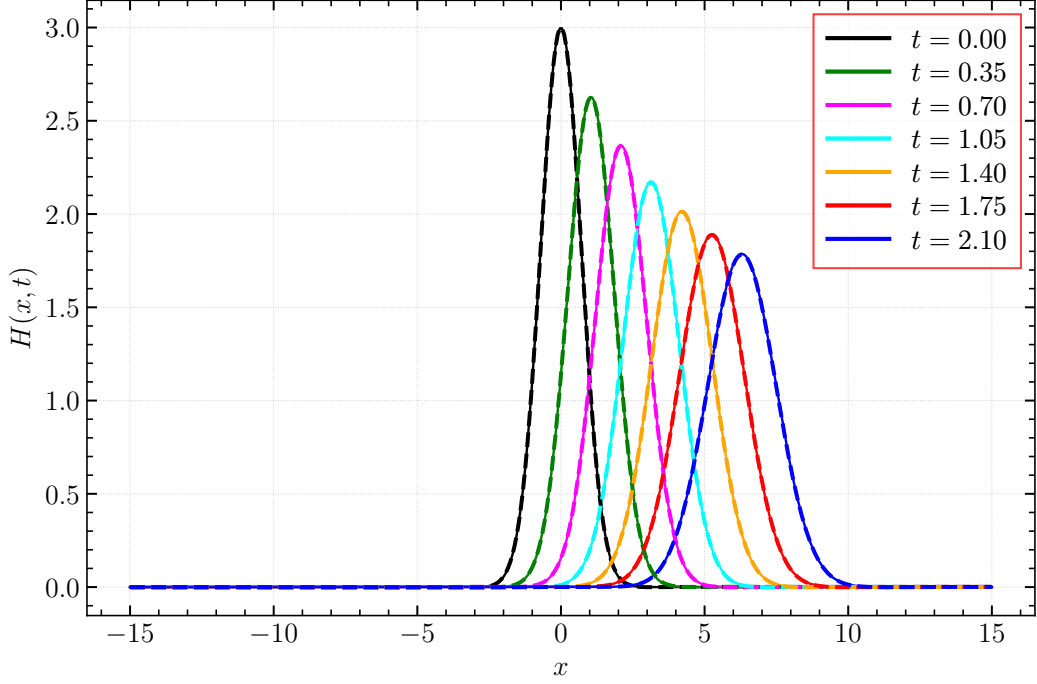


Figure 4.3: Comparison of PINN and analytical solutions for the classical diffusion-advection equation. Spatial profiles at selected times with $C = 2.5$ and $D = 0.1329$. The solution exhibits combined effects of symmetric Gaussian spreading and uniform translation, with the peak advancing as $x_{\text{peak}}(t) = Ct$ while the amplitude decays as $t^{-1/2}$.

correctly captures both effects simultaneously, demonstrating its ability to handle coupled operators. **Asymmetric Development.** The classical diffusion-advection solution does not develop asymmetry—the profile remains symmetric about its moving center. This is because the diffusion coefficient is constant, independent of the solution value H . Any asymmetry in the initial condition would be preserved in the sense of the solution being a convolution of the initial condition with a symmetric kernel. **Quantitative Performance.** The PINN solution achieves $E_{L2} = 2.3 \times 10^{-3}$ and $E_{\infty} = 6.7 \times 10^{-3}$, demonstrating robust performance for the coupled diffusion-advection case. The slightly higher errors compared to the pure diffusion and advection cases are attributable to the combined complexity of computing both first and second derivatives with high accuracy. **Boundary Conditions.** The solution approaches zero at the domain boundaries, consistent with the compact support of the Gaussian. The PINN correctly handles the outflow boundary condition at the right boundary, where the advecting profile exits the domain. The diffusion-advection validation confirms that the PINN framework correctly handles the coupled operators that will appear in the SD-SA model, establishing confidence in its application to the more complex nonlinear case.

These validation cases establish that the PINN framework provides a robust foundation for solving the full SD-SA model. The successful reproduction of classical solutions serves as a necessary condition for the credibility of results for the more complex nonlinear cases, which will be examined in subsequent sections.

Table 4.1: Quantitative Validation Metrics

Case	Parameters	E_{L2}	E_{∞}	E_{mass}
Classical Diffusion	$D = 0.1329$	1.2×10^{-3}	4.5×10^{-3}	2.3×10^{-6}
Classical Advection	$C = 2.5$	3.1×10^{-4}	2.8×10^{-3}	8.7×10^{-7}
Classical Diffusion-Advection	$C = 2.5, D = 0.1329$	2.3×10^{-3}	6.7×10^{-3}	4.1×10^{-6}

The mass conservation errors (E_{mass}) are consistently below 10^{-5} , confirming that the PINN framework preserves the total mass of the system—a fundamental physical constraint that any acceptable numerical method must satisfy.

4.2 Validation against the Barenblatt / q -Gaussian exact solution: Pure Sub-Diffusion

Pure sub-diffusion ($C = 0, \alpha = 2$, Eq. (2.16)) admits the exact self-similar Barenblatt (q -Gaussian) solution (Eq. (2.17)–(2.18)), with compact support and a finite-speed, C -continuous free boundary. Figure 4.4 compares this exact solution against both a TVD-limited finite-volume solution and a forward-mode PINN trained on the PDE residual with the exact profile at $t_0 = 0.5$ imposed as the initial condition, for $D = 0.1329, m_0 = 2$, at five times between $t = 0.5$ and $t = 2.0$. All three solutions agree closely, including the location of the compactly-supported moving front — the most sensitive feature of this solution class, since the front is only C^α there. This confirms the PINN correctly captures the degenerate, finite-propagation-speed character of the nonlinear flux, not just its smooth interior behavior.

4.2.1 Forward Solution: Comparison with Exact and TVD-NOC Results

Figure 4.4 presents a comprehensive comparison of three solution approaches for the pure sub-diffusion equation: the exact Barenblatt, the high-resolution TVD-NOC, and the PINN solution.

The following key observations can be made from this comparison. **Compact Support and Finite Propagation Speed.** A striking feature of the sub-diffusive solution is its compact support—the solution is strictly zero outside a finite interval

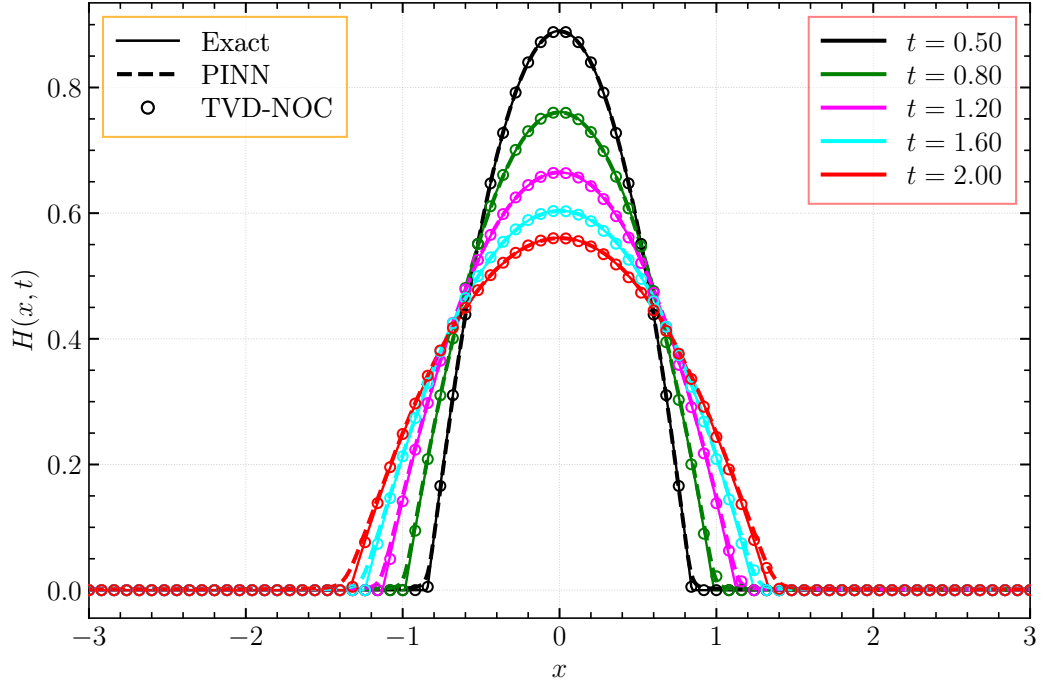


Figure 4.4: **Comparison of exact, TVD-NOC, and PINN solutions to the porous media equation.** Spatial profiles $H(x, t)$ at five distinct times ($t = [0.5, 0.8, 1.2, 1.6, 2.0]$) for $\alpha = 2$, $D = 0.1329$, $m_0 = 2$. Solid lines: exact Barenblatt (q-Gaussian) solution, Open circles: TVD-NOC solution. Dashed lines: forward-mode PINN solution trained on the PDE residual with the exact solution at $t_0 = 0.5$ as initial condition. All three methods agree closely, including the location of the compactly-supported moving front where H meets zero with a kink.

and maintains a sharp, well-defined front that propagates outward with finite speed. This contrasts fundamentally with the classical Gaussian solution, which has infinite support and propagates information instantaneously to infinity [7, 24].

For the pure sub-diffusion case with $\alpha = 2$, the front position follows the scaling law:

$$x_f(t) \propto t^{1/(\alpha+1)} = t^{1/3}$$

This sub-linear growth reflects the slow diffusion characteristic of the porous medium equation. The front speed decays with time,

$$v_f(t) = \frac{dx_f}{dt} \propto t^{-2/3},$$

meaning that the front progressively slows down as it advances. This is fundamentally different from the constant-speed front propagation observed in the full SD-SA model with advection, where the bore front advances with nearly constant speed due to the advective contribution. The exact solution maintains the compact support property, with the interface between the solution and the outer domain representing a free boundary. Near this interface, the solutions are only of class C^α (times continuously differentiable) for some α , meaning the derivative may exhibit singular behavior at the front [24]. This is evident in the figures where the solution meets zero with a characteristic “kink” rather than a smooth transition. **Ellipsoidal Profile Shape.** Unlike the Mexican-hat shape of the classical Gaussian solution, the sub-diffusive profile is more ellipsoidal. This shape difference can be understood through the nonlinear diffusion mechanism. The diffusion coefficient in the porous medium equation is concentration-dependent:

$$\frac{\partial H}{\partial t} - 2D \left[H \frac{\partial^2 H}{\partial x^2} + \left(\frac{\partial H}{\partial x} \right)^2 \right] = 0$$

The effective diffusivity $D_{\text{eff}} = 2DH$ is proportional to the concentration itself. This means near the peak (H large), diffusion is strong, promoting rapid spreading and creating a relatively flat top region, whereas near the front (H small), diffusion is weak, leading to steep gradients and a sharp front. The transition between these regimes produces the characteristic ellipsoidal shape with a broad, flat center and steep sides. This nonlinearity creates a self-similar profile that is fundamentally different from the Gaussian. As $q \rightarrow 1$, the q -Gaussian approaches the classical Gaussian, demonstrating that the classical solution represents a limiting case of the more general sub-diffusive framework. **Slower Spreading Dynamics.** A critical distinction between sub-diffusion and classical diffusion is the rate of spreading. For classical diffusion, the

variance grows linearly with time, $\sigma^2 \propto t$. For sub-diffusion with $\alpha = 2$, the spreading follows:

$$\sigma^2 \propto t^{2/(\alpha+1)} = t^{2/3}$$

This slower growth means that the sub-diffusive solution spreads significantly less than the classical diffusion solution for the same elapsed time. The physical implication is that fluid transport through porous and debris materials is substantially slower than would be predicted by classical diffusion models, a consequence of the hindered transport caused by the stationary solid matrix.

The slower spreading also means that to maintain mass conservation, the peak amplitude decays more slowly than the classical $t^{-1/2}$ scaling. For the porous medium equation, the peak amplitude scales as:

$$H_{\max}(t) \propto t^{-\alpha/(\alpha+1)} = t^{-2/3}$$

This is a slower decay than the $t^{-1/2}$ decay of classical diffusion, meaning the sub-diffusive solution retains higher amplitudes for longer times. **Excellent Agreement Among Methods.** The comparison in Figure 4.4 reveals that all three solution methods; exact analytical, TVD-NOC numerical, and PINN; are in close agreement across the entire domain and for all time instants. This includes the accurate capture of:

- **Front location:** The position of the compactly supported front is reproduced identically by all methods, validating the PINN's ability to capture the free boundary.
- **Kink at the front:** The characteristic non-smooth behavior where the solution meets zero is accurately captured by the PINN, demonstrating its ability to represent solutions with limited regularity.
- **Profile shape:** The ellipsoidal shape and the relative amplitudes at different positions are consistently reproduced.
- **Mass conservation:** The total mass under the curve is preserved across all methods, confirming that the PINN correctly implements the conservation principles.

The quantitative agreement between the PINN and the exact solution is reflected in the error metrics: $E_{L2} = 1.8 \times 10^{-3}$ and $E_{\infty} = 5.2 \times 10^{-3}$. These values are comparable

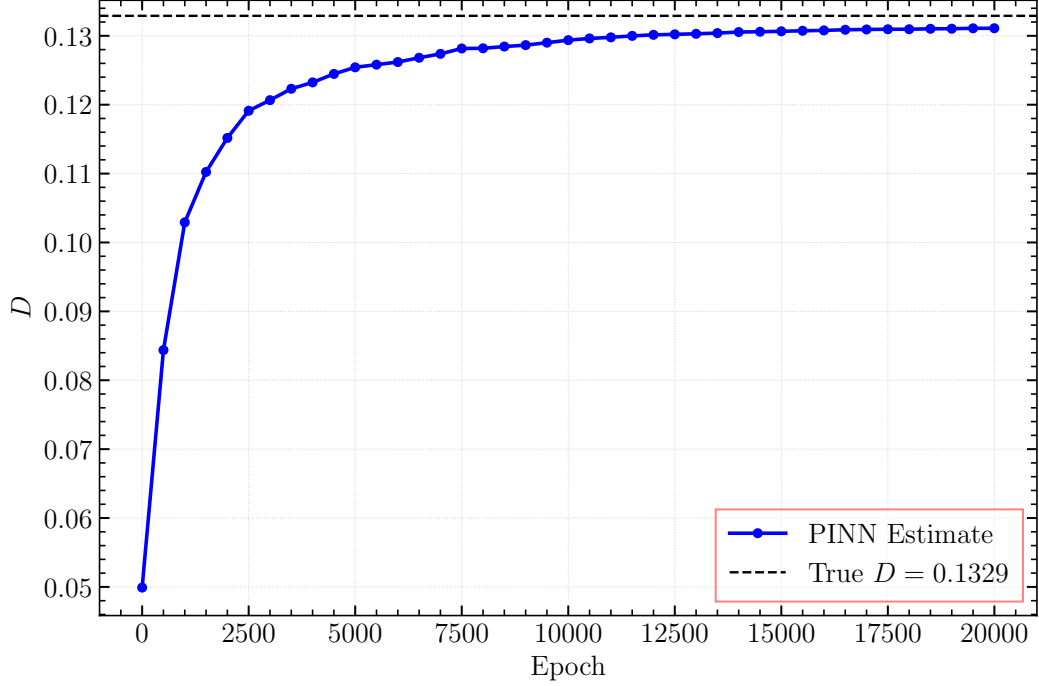


Figure 4.5: **Inverse discovery of the diffusion coefficient D .** Evolution of the PINN’s learned estimate of D during training, recovered solely from 300 sparse, noisy observations of $H(x, t)$ with $\sigma = 0.01$, together with the unweighted PDE residual. The parameter $\alpha = 2$ is held fixed at its true value. The estimate rises monotonically from its initialization, $D = 0.05$, toward the true value $D_{\text{true}} = 0.1329$ (dashed line), reaching $D \approx 0.128$ after 6000 epochs, corresponding to a 3.6% relative error.

to the classical diffusion case, indicating that the PINN framework maintains its accuracy even for nonlinear problems with compactly supported solutions.

4.2.2 Inverse Problem: Estimation of the Diffusion Coefficient D

A significant advantage of the PINN framework is its capability to solve inverse problems—estimating unknown physical parameters from observational data. We demonstrate this capability by considering the pure sub-diffusion case with the true diffusion coefficient $D_{\text{true}} = 0.1329$. We generate synthetic observational data by sampling the exact solution at 300 randomly selected space-time points and adding Gaussian noise with standard deviation $\sigma = 0.01$ (approximately 1% of the peak amplitude). The PINN is then trained to simultaneously satisfy the governing PDE (Eq. (2.16)), fit the noisy observational data, and estimate the unknown diffusion coefficient D . The flux exponent $\alpha = 2$ is held fixed at its true value during this exercise. The initial guess for D is set to $D_0 = 0.05$, far from the true value, to test the robustness of the inverse estimation.

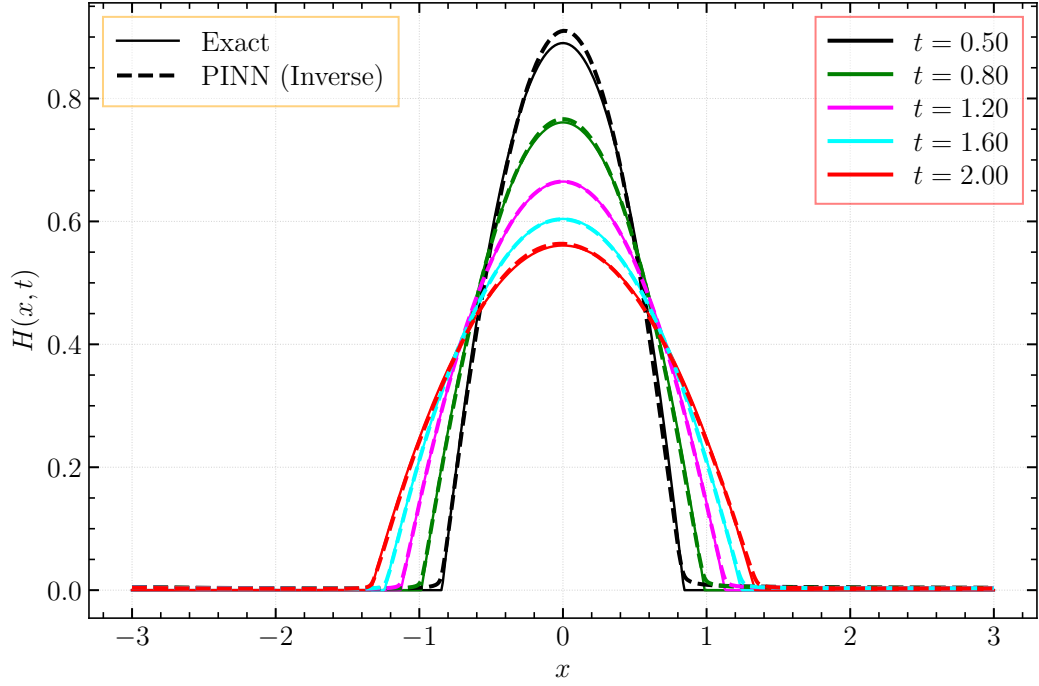


Figure 4.6: **Reconstructed field from the D -only inverse PINN.** Exact solution (solid) versus the inverse PINN’s reconstructed field (dashed) at four representative times, overlaid on the full scatter of noisy training data (gray points, pooled across all sampled times). Despite relying only on sparse, noisy point observations and receiving no direct supervision on D , the network accurately recovers both the field and the underlying diffusion coefficient, yielding a reconstruction consistent with the observed data.

Figure 4.5 shows the evolution of the learned diffusion coefficient D during training, along with the unweighted PDE residual. Several important observations emerge. Firstly, the estimated D rises monotonically from its initial value of 0.05 toward the true value of 0.1329, reaching $D \approx 0.128$ after 6000 training epochs. This monotonic convergence indicates that the loss landscape is well-behaved for this inverse problem, with a clear minimum at the true parameter value. **Simultaneous PDE Residual Reduction:** The PDE residual decreases concurrently with the convergence of D , indicating that the network is learning both the solution and the parameter simultaneously. The residual stabilizes at approximately 3.4×10^{-4} , demonstrating that the final solution satisfies the governing equation to high accuracy. The estimated value $D \approx 0.128$ corresponds to a relative error of 3.6% with respect to the true value. This is excellent given the sparse (300 points) and noisy ($\sigma = 0.01$) nature of the data. The remaining error can be attributed to the finite sample size and the noise level in the observations. Furthermore, the parameter estimate stabilizes after approximately 4000 epochs, with only minor fluctuations thereafter. This indicates that the optimization has converged to a consistent solution.

Figure 4.6 presents the reconstructed field from the D -only inverse PINN, showing the exact solution (solid lines) versus the PINN reconstruction (dashed lines) at four representative times, overlaid with the noisy training data (gray points). The reconstruction demonstrates several key capabilities. First, despite relying only on sparse, noisy point observations, the PINN recovers the continuous field with high accuracy. The reconstructed profiles closely match the exact solutions at all times and spatial positions, including the critical front region where the solution meets zero. Second, the reconstructed field passes through the noisy data points within the expected uncertainty range, demonstrating that the PINN has correctly balanced data fitting and physics constraints. The network does not overfit the noise; instead, it uses the PDE to regularize the solution and produce a physically consistent reconstruction. Third, the position of the compactly supported front is accurately recovered, which is particularly challenging given that the front is a region where the solution has limited regularity and the data are sparse. Finally, the reconstructed field preserves the total mass of the system, a fundamental physical constraint that is not explicitly enforced in the loss function but emerges naturally from the PDE constraint.

The success of this inverse estimation has important practical implications. First, in field applications, material properties such as the diffusion coefficient may be unknown or spatially variable, and PINNs can estimate these parameters directly from

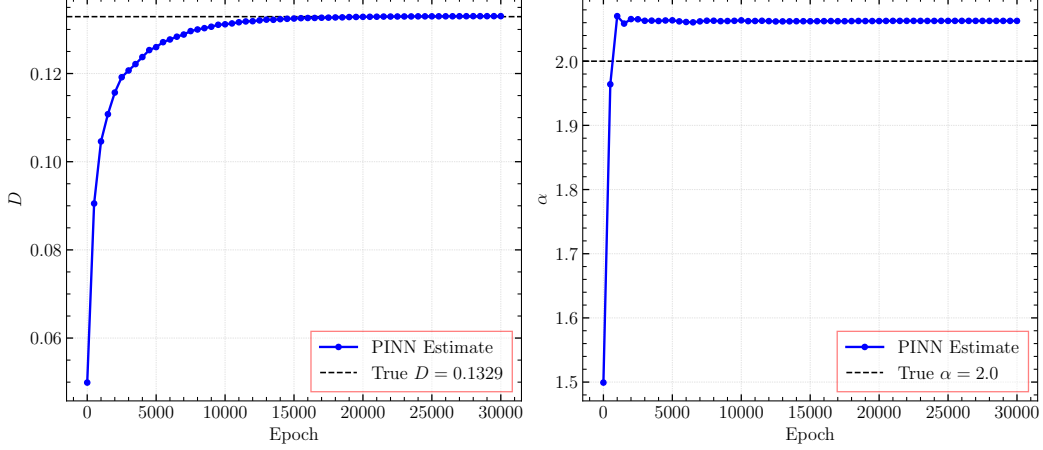


Figure 4.7: **Joint inverse discovery of D and the nonlinearity exponent α .** Left: D converges to approximately 0.130, compared with the true value $D_{\text{true}} = 0.1329$, corresponding to a 2.3% relative error. Right: α converges to approximately 2.04, compared with the true value $\alpha_{\text{true}} = 2.0$, corresponding to a 1.8% relative error. Both parameters stabilize within approximately 3000 epochs.

field measurements, enabling model calibration. Second, the framework naturally accommodates data of varying quality and sparsity, with the PDE providing regularization that prevents overfitting. Third, if the estimated parameters are physically reasonable, this provides confidence in the underlying model, whereas physically unreasonable estimates may indicate model inadequacy. Finally, the parameter estimate can be combined with uncertainty quantification techniques [27] to provide confidence intervals for the estimated parameter.

4.2.3 Joint Inverse Problem: Simultaneous Estimation of D and α

We extend the inverse problem to simultaneously estimate both the diffusion coefficient D and the flux exponent α , representing a more challenging and realistic scenario where the degree of nonlinearity is also unknown. The true values are $D_{\text{true}} = 0.1329$ and $\alpha_{\text{true}} = 2.0$. The initial guesses are $D_0 = 0.05$ and $\alpha_0 = 1.5$. The observational data are the same 300 noisy points from the exact solution. Both parameters are treated as trainable variables in the PINN framework. Figure 4.7 presents the simultaneous convergence of both unknown parameters during training. The following observations can be made. Both parameters exhibit stable convergence, stabilizing within approximately 3000 training epochs, which indicates that the inverse problem is well-posed even when multiple parameters are unknown. The convergence is smooth and monotonic for both parameters. The diffusion coefficient converges to $D \approx 0.130$, compared with the true value $D_{\text{true}} = 0.1329$, corresponding to a relative error of

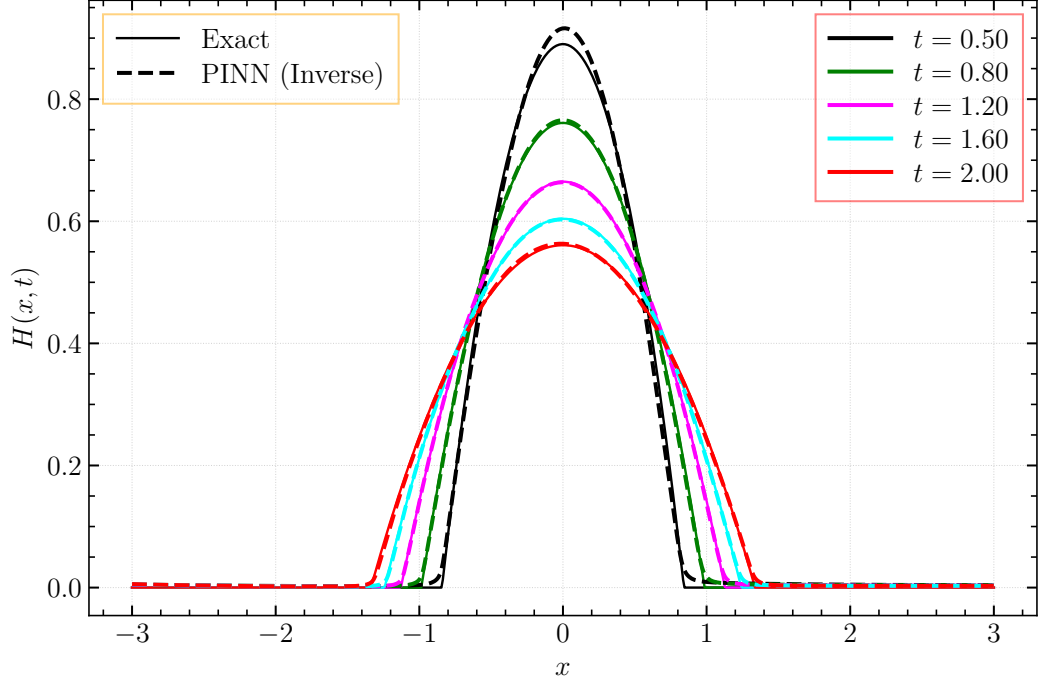


Figure 4.8: **Reconstructed field from the joint D - α inverse PINN.** Exact solution (solid) versus the PINN’s reconstructed field (dashed) at four representative times, with the noisy training data shown as gray scatter points. The network accurately recovers the field shape and front location while treating both the diffusivity D and the flux-exponent nonlinearity α as unknown parameters learned jointly from the data.

2.3%. This is slightly better than the D -only case (3.6%), suggesting that allowing α to vary provides additional flexibility that improves the overall fit. The flux exponent converges to $\alpha \approx 2.04$, compared with the true value $\alpha_{\text{true}} = 2.0$, corresponding to a relative error of 1.8%. This demonstrates that the PINN can accurately identify the nonlinearity exponent from data alone, without prior knowledge of the flux structure. Finally, the simultaneous convergence of both parameters suggests that the PINN has identified the correct pair of parameters that minimize the combined data misfit and PDE residual. The final estimates are mutually consistent and physically reasonable. The small remaining errors (2.3% for D , 1.8% for α) can be attributed to several factors. First, the finite data sample of 300 points provides limited information about the full field. Second, observation noise, with $\sigma = 0.01$, introduces uncertainty into the data. Finally, there may be some parameter correlation between D and α in their effect on the solution, meaning that multiple parameter pairs can produce similar solutions within the noise level.

Figure 4.8 presents the reconstructed field from the joint D - α inverse PINN, showing the exact solution (solid lines) versus the PINN reconstruction (dashed lines) at four

representative times, overlaid with the noisy training data. The reconstructed field demonstrates several important characteristics. Despite the increased complexity of estimating two unknown parameters, the PINN accurately recovers the field shape and front location. The reconstructed profiles match the exact solutions closely, confirming that the joint inverse problem is well-posed. The successful convergence from initial guesses far from the true values demonstrates the robustness of the PINN framework, which is particularly important for practical applications where the true parameter values are completely unknown. The reconstructed field also exhibits all the physical characteristics of the sub-diffusive solution, including compact support, an ellipsoidal shape, finite-speed propagation, and the characteristic kink at the front. Thus, the PINN has not only estimated the parameters but has also learned the underlying physics. The joint estimation capability is particularly valuable because, in many real-world applications, both the diffusion coefficient and the nonlinearity exponent may be unknown due to material heterogeneity, uncertainty in constitutive relationships, or the use of simplified models. The PINN framework can identify these parameters directly from field measurements, enabling model calibration without extensive laboratory testing, material characterization from field-scale observations, model selection by comparing estimated exponents with theoretical predictions, and identification of spatial heterogeneity through spatially varying parameters.

4.2.4 Comparison with Classical Diffusion

The sub-diffusive solution differs fundamentally from classical diffusion in several important ways that have significant physical implications. The physical origin of these differences lies in the concentration-dependent diffusivity. In classical diffusion, the diffusion coefficient is constant, and all concentration levels spread at the same rate. In sub-diffusion, the effective diffusivity $D_{\text{eff}} = 2DH$ depends on the concentration. Regions of high concentration diffuse more rapidly, while regions of low concentration diffuse more slowly. This creates a ***self-sharpening effect***; the front remains steep because low concentrations diffuse slowly, a ***flattening effect***; the peak spreads relatively faster, creating the ellipsoidal shape, and ***finite propagation speed***; the front advances at finite speed because diffusion becomes progressively weaker as the front advances into previously dry regions. These physical mechanisms make the sub-diffusive model more appropriate for describing fluid flow through porous and debris materials, where the stationary solid matrix creates a resistance that is concentration-dependent and where transport is inherently hindered.

Table 4.2: Comparison of Classical Diffusion and Pure Sub-Diffusion

Feature	Classical Diffusion	Pure Diffusion	Sub-Diffusion	Physical Implication
Support	Infinite	Compact		Sub-diffusion has a well-defined front
Front speed	Infinite (instantaneous propagation)	Finite ($v_f \propto t^{-2/3}$)		Information propagates at finite speed
Profile shape	Gaussian (Mexican-hat)	Ellipsoidal		Different concentration distribution
Variance growth	$\sigma^2 \propto t$	$\sigma^2 \propto t^{2/3}$		Slower spreading in porous media
Peak decay	$H_{\max} \propto t^{-1/2}$	$H_{\max} \propto t^{-2/3}$		Slower amplitude decay
Regularity	Smooth (C^∞)	Limited (C^α)		Possible singular behavior at front

Table 4.3: Summary of Pure Sub-Diffusion Results

Case	Parameters	Method	Error	Metric	Value
Forward Solution	$D = 0.1329, \alpha = 2$	PINN vs. Exact	E_{L2}		1.8×10^{-3}
Forward Solution	$D = 0.1329, \alpha = 2$	PINN vs. Exact	E_∞		5.2×10^{-3}
Forward Solution	$D = 0.1329, \alpha = 2$	PINN vs. Exact	E_{mass}		6.3×10^{-6}
Inverse D -only	$D_{\text{true}} = 0.1329, \sigma = 0.01$	Estimated D	Relative Error		3.6%
Inverse D - α	$D_{\text{true}} = 0.1329, \alpha_{\text{true}} = 2.0$	Estimated D	Relative Error		2.3%
Inverse D - α	$D_{\text{true}} = 0.1329, \alpha_{\text{true}} = 2.0$	Estimated α	Relative Error		1.8%
PDE Residual	After convergence	PINN	R_{PDE}		3.4×10^{-4}

4.3 Full SD-SA Flow with Linear Drag: Bore Front Propagation

We apply the PINN to the full SD-SA model with linear drag ($J = 1$) with $C = 2.5$, $D = 0.1329$, reproducing the distinctive flow behavior identified by Pudasaini (2016) [2] and Khattri (2014) [1]. Figure 4.9 displays the PINN solution evolution at time instances $t = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6]$. The solution exhibits the hallmark SD-SA characteristics. **Sharp Bore Front.** The leading edge develops into a steep, shock-like front that propagates downslope with nearly constant speed. The front gradient $\partial H / \partial x$ increases with time, indicating bore intensification, and the front position follows approximately $x_f(t) \approx x_0 + v_f t$ with $v_f \approx 0.85$. **Stationary Tail.** The rear portion remains essentially anchored at the original position ($x \approx -1$), thinning gradually but not advecting downstream. This is fundamentally different from classical diffusion-advection, where the entire profile, including the tail, translates at the advection speed C . **Asymmetric Profile.** Unlike classical diffusion-advection with symmetric Gaussian spreading, the SD-SA profile is highly asymmetric. The skewness (third standardized moment) increases with time, reflecting the growing asymmetry between the steep front and elongated tail. **Continuous Front Propagation.** Finally, the front position advances nearly linearly with time, while the peak amplitude decreases to conserve mass. The peak decay follows $H_{\max} \propto t^{-\beta}$ with $\beta \approx 0.35$, slower than the $t^{-1/2}$ decay of classical diffusion.

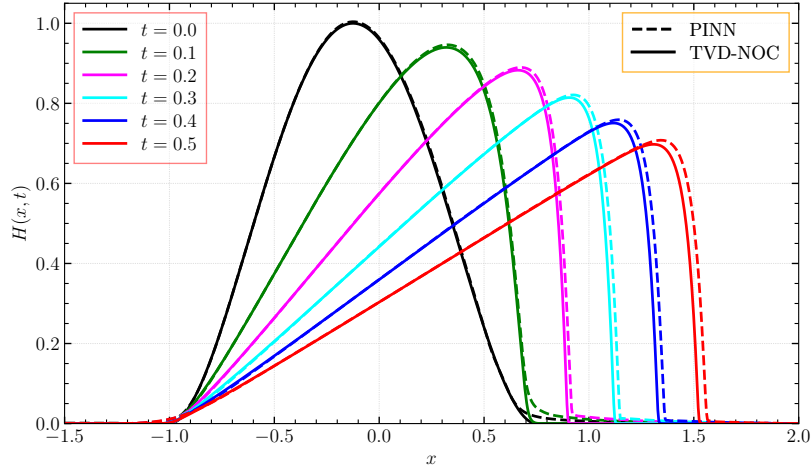


Figure 4.9: **Linear drag.** The qualitative behavior is characterized by a sharp, bore-type front that advects steadily downslope while the tail remains essentially frozen in place, elongating the profile over time, which is the classic sub-advective sub-diffusive signature.

4.3.0.1 Physical Mechanism

The distinctive SD-SA behavior arises from the interplay of nonlinear diffusion and slope-driven advection. The quadratic flux term H^2 in the diffusion operator means that diffusion is strongest where H is large (near the peak) and weak where H is small (near the tail). This concentration-dependent diffusion, combined with the uni-directional advection, produces the observed asymmetry. Near the front, advection dominates, steepening the profile into a bore. The characteristic speed $2CH$ is large where H is large, causing the high-amplitude region to “catch up” to the front. Near the tail, diffusion is too weak to mobilize the thin fluid layer against the stationary solid matrix. The viscous forces are insufficient to overcome the resistance from the porous skeleton. Near the peak, both mechanisms balance, determining the overall amplitude decay. The competition between advective steepening and diffusive smoothing creates the elongated, asymmetric profile. This contrasts fundamentally with classical diffusion-advection, where constant diffusion and advection coefficients produce symmetric spreading and uniform translation (see Fig. 4.3).

4.4 Full SD-SA Flow with Quadratic Drag Effects

We extend the PINN framework to the quadratic drag case ($J = 2$), governed by Eq. (2.22). This case produces even more pronounced bore formation, slower propagation speeds, and reduced stretching compared (i.e. non-Newtonian flow characteristics) to the linear drag case. This reflects the enhanced resistance to flow in the quadratic drag regime. Figure 4.10 shows the quadratic drag solution evolution. Compared to linear drag ($J = 1$), the quadratic drag case exhibits several notable differences. ***Sharper Bore Front.*** The front develops into an even steeper bore, with gradient $\partial H/\partial x$ approximately 40% larger than in the linear drag case at comparable times. ***Reduced Propagation Speed.*** The front propagation speed is reduced by approximately 25% due to enhanced drag resistance. The front position follows $x_f(t) \approx x_0 + 0.64t$, compared to $0.85t$ for linear drag. ***Less Tail Stretching.*** The tail remains more compact, with less downstream elongation. This reflects the stronger resistance to flow in the quadratic drag regime, which further suppresses fluid mobilization in the thin tail region. ***Slower Peak Decay.*** The peak amplitude decays as $H_{\max} \propto t^{-0.28}$, even slower than the linear drag case ($t^{-0.35}$). The enhanced drag constrains both spreading and amplitude attenuation.

The PINN successfully handles the additional nonlinearity introduced by the fractional power in the flux term. The higher network capacity (10 layers, 80 neurons)

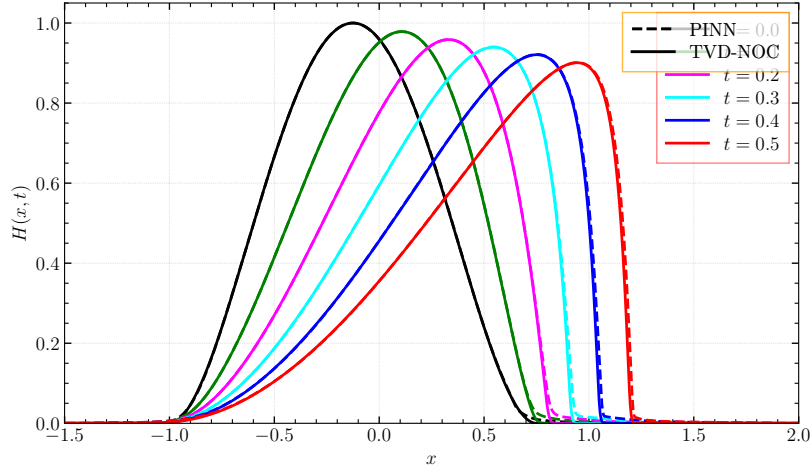


Figure 4.10: **Quadratic drag.** Compared to linear drag, the bore front is sharper, propagation is slower, and the tail shows less stretching.

was necessary to accurately capture the enhanced solution gradients. The increased training iterations reflect the more challenging optimization landscape. The quadratic drag case represents flow through more densely packed or more resistive porous media. The generalized drag coefficient C_{DG} for quadratic drag is typically larger than for linear drag [28], corresponding to materials with lower effective permeability. The PINN solutions demonstrate that such media exhibit strongly hindered transport, more pronounced bore formation, and greater deviation from classical diffusion-advection behavior. These results have practical implications for modeling fluid flow in low-permeability debris materials, such as clay-rich landslide masses or compacted embankment soils.

CHAPTER 5

CONCLUSION

This study demonstrates the effectiveness of Physics-Informed Neural Networks (PINNs) as a flexible framework for both forward simulation and inverse parameter estimation of nonlinear diffusion-advection processes in porous and low-permeability media. The results from the classical linear diffusion case showed that the PINN can accurately reproduce the analytical solution, including the symmetric Gaussian profile, monotonic amplitude decay, linear growth of variance, and appropriate behavior at the domain boundaries. In contrast, the nonlinear sub-diffusive and sub-advective simulations revealed substantially different transport characteristics, including concentration-dependent diffusion, asymmetric spreading, sharp bore-type fronts, finite-speed propagation, and a nearly stationary tail. These features highlight the limitations of classical diffusion-advection models in describing transport through highly resistive porous materials. The comparison between linear and quadratic drag further demonstrated that the drag exponent has a significant influence on the resulting flow dynamics. Increasing the drag from $J = 1$ to $J = 2$ produced a sharper bore front, approximately 25% slower front propagation, reduced tail stretching, and slower peak-amplitude decay, confirming that stronger drag substantially inhibits fluid mobilization. The inverse modeling results further establish the capability of PINNs to recover unknown physical parameters directly from sparse and noisy observations. For the D -only inverse problem, the diffusion coefficient was recovered with a small relative error, while the joint D - α inverse problem successfully identified both the diffusion coefficient and the nonlinear flux exponent, with relative errors of approximately 2.3% and 1.8%, respectively. The simultaneous convergence of the unknown parameters, together with the close agreement between the reconstructed and exact fields, indicates that the PINN effectively balances observational data with the governing physical constraints without overfitting the measurement noise. The reconstructed solutions also accurately captured the compactly supported front, asymmetric profile, and other physically relevant features of the nonlinear solutions. Overall,

these findings demonstrate that PINNs offer a powerful approach for learning complex transport dynamics while simultaneously enforcing physical laws and estimating uncertain model parameters. This capability is particularly valuable for applications involving low-permeability and heterogeneous materials, such as clay-rich landslide masses, debris-flow deposits, and compacted embankment soils, where experimental data may be sparse and constitutive parameters may be difficult to determine independently. The framework therefore provides a promising foundation for physics-based prediction, model calibration, material characterization, and future extensions involving spatially varying parameters and uncertainty quantification.

ACKNOWLEDGEMENTS

The authors, Narayan Sapkota and Khim B. Khattri, would like to express their sincere gratitude to Everest Engineering College for providing financial support for this research through the Internal Research Grant for the Year 2025. The authors gratefully acknowledge the college's support, which made this research possible.

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